Moderately exponential approximation

Bridging the gap between exact computation and polynomial approximation

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1. Quick recalls
2. Moderately exponential approximation
3. Techniques for moderately exponential approximation
4. Some questions

Approximation ratio

Approximation ratio of an approximation algorithm $\mathtt{A}$

$$\rho(\mathtt{A}, I, S) = \frac{\text{value of the solution } S \text{ computed by } \mathtt{A} \text{ on } I}{\text{optimal value}}$$

The closer the ratio to 1, the better the performance of $\mathtt{A}$

Inapproximability

Inapproximability result

A statement that a problem is not approximable within ratios better than some approximability level unless something very unlikely happens in complexity theory

- $\boldsymbol{P} = \boldsymbol{NP}$
- *Disproval of the* ***ETH***
- ...

ETH

SAT or one of its mates cannot be solved to optimality in subexponential time

Examples of inapproximability

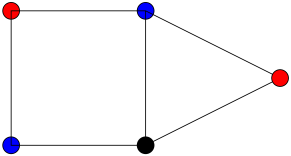
- MAX INDEPENDENT SET or MAX CLIQUE inapproximable within ratios $\Omega\left(n^{-1}\right)$
- MIN VERTEX COVER within ratios smaller than 2
- MIN SET COVER within ratios $o(\log n)$
- MIN TSP within better than exponential ratios
- MIN COLORING within ratios $o(n)$
- ...

Guiding thread of the talk

The MAX INDEPENDENT SET problem

MAX INDEPENDENT SET

Given a graph $G(V,E)$ we look for a maximum size $V' \subseteq V$ such that $\forall (v_i, v_j) \in V' \times V', (v_i, v_j) \notin E$



Exact computation with worst-case bounds (1)

*Determine an optimal solution for an **NP**-hard problem with provably non trivial worst-case time-complexity*

For MAX INDEPENDENT SET

- Exhaustively generate any subset of V and get a maximum one among those that are independent sets: $O\left(2^n\right)$ (trivial exact complexity)
- Find all the maximal independent sets of the input graph: $O\left(1.4422^n\right)$ (Moon & Moser (1965))

Exact computation with worst-case bounds (2): pruning the search tree

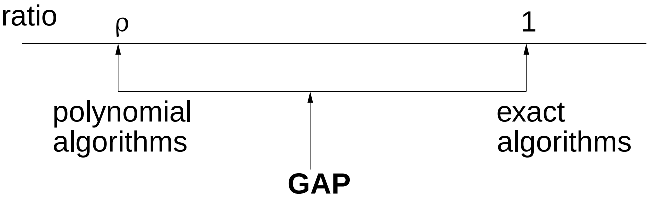
(a) 1 vertex fixed

(b) $\geqslant$ 4 vertices fixed

$$T(n) \leqslant T(n-1) + T(n-4) + p(n) \simeq O\left(1.385^n\right)$$

$\longrightarrow$ Numerous subsequent improvements

A basic question (goal = max)



- What about **GAP**?
- Why not taking advantage of the power of modern computers?
- For realistic values of n, 1.1^n is not so "worse" than, say, n^5

The key issue

Approximate optimal solutions of NP-hard problems within ratios “forbidden” to polynomial algorithms and with worst-case complexity provably better than the complexity of an exact computation

Do it

- For some forbidden ratio
- For any forbidden ratio

Generate a small number of candidates (1)

The key-idea

Generate a small number of candidate solutions (polynomially complete them, if necessary and possible) and return the best among them

Generate a small number of candidates (2): MAX INDEPENDENT SET

- Generate all the $\sqrt{n}$-subsets of V
- If one of them is independent, then return it
- Else return a vertex at random

Approximation ratio: $n^{-1/2}$ (impossible in polynomial time)

Worst-case complexity: $O\left(\binom{n}{\sqrt{n}}\right) \leqslant O\left(2^{\sqrt{n}\log n}\right)$

Subexponential

Generate a small number of candidates (2): works also for ...

- MIN INDEPENDENT DOMINATING SET (Bourgeois, Escoffier & P (2010))
- CAPACITATED DOMINATING SET (Cygan, Pilipczuk & Wojtaszczyk (2010))

Divide & approximate (1)

The key-idea

Optimally solve a problem in a series of (small) sub-instances of the initial instance

- Appropriately split the instance in a set of sub-instances (whose sizes are functions of the ratio that is to be achieved)
- Solve the problem in this set
- Compose a solution for the initial instance using the solutions of the sub-instances

Divide & approximate (2): MAX INDEPENDENT SET

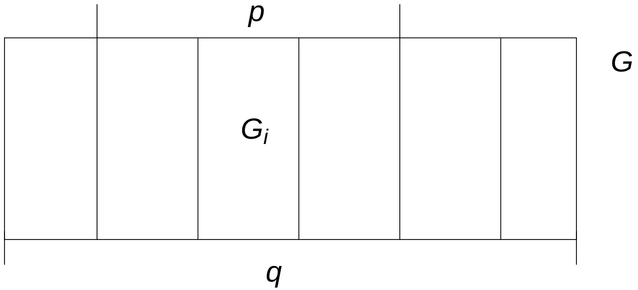
Theorem

Assume that an optimal solution for MAX INDEPENDENT SET *can be found in* $O\left(\gamma^{n}\right)$
Then, for any fixed p, q, $p < q$, *a* (p/q)*-approximation can be computed in* $O\left(\gamma^{\frac{p}{q}n}\right)$

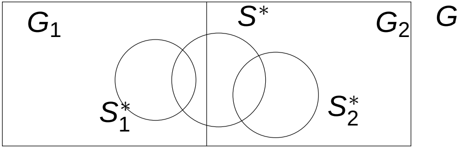
It works for any problem defined upon a hereditary property

Divide & approximate (3)

- Build the unions of all the p subgraphs in $\{G_1, \ldots, G_q\}$ among q
- Take the best among these $\binom{q}{p}$ solutions



Divide & approximate (4): example for $p/q = r = 1/2$



$$|S^*| \leqslant |S_1^*| + |S_2^*| \leqslant 2\max\{|S_1^*|, |S_2^*|\} \Longrightarrow \frac{\max\left\{\left|S_1^*\right|, \left|S_2^*\right|\right\}}{|S^*|} \geqslant \frac{1}{2}$$

Complexity: $O\left(\gamma^{n/2}\right)$

Divide & approximate (5): works also for ...

If $O\left(\gamma^{n}\right)$ the complexity for MAX INDEPENDENT SET

- MIN VERTEX COVER: $(2-r)$-approximation in $O\left(\gamma^{rn}\right)$, for any r (Bourgeois, Escoffier & P (2011))
- MAX CLIQUE: r-approximation in $O\left(\gamma^{r\Delta}\right)$ (Δ the maximum degree of the input-graph), for any r (Bourgeois, Escoffier & P (2011))
- MAX SET PACKING: r-approximation in $O\left(\gamma^{rn}\right)$, for any r (Bourgeois, Escoffier & P (2011))
- MAX BIPARTITE SUBGRAPH: r-approximation in $O\left(\gamma^{2rn}\right)$, for any r (Bourgeois, Escoffier & P (2011))

Approximately pruning the search tree (1)

The key idea

Perform a branch-and-cut by allowing a “bounded error” in order to accelerate the algorithm (i.e., make the instance-size decreasing quicker than in exact computation by keeping the produced error “small”)

Approximately pruning the search tree (2): MAX INDEPENDENT SET

1. If $\Delta(G) \leqslant 7$, then approximate MAX INDEPENDENT SET polynomially;
2. else, branch on a vertex v with $d(v) \geqslant 8$ and either take it, remove its neighbors and two more vertices v_i, v_j such that $(v_i, v_j) \in E$, or do not take it

Approximately pruning the search tree (3): MAX INDEPENDENT SET (cont.)

Theorem

The above algorithm computes an $\frac{1}{2}$-approximation for MAX INDEPENDENT SET *in time* $O\left(1.185^n\right)$

Approximately pruning the search tree (4): MAX INDEPENDENT SET (cont.)

- If $\Delta(G) \leqslant 7$, approximation ratio $\frac{1}{2}$ (ratio $\frac{5}{\Delta(G)+3}$, (Berman & Fujito (1985)))
- If $\Delta(G) \geqslant 8$, we make an "error" of at most 1 vertex per vertex introduced in the solution (ratio $\frac{1}{2}$)

Complexity

$$T(n) \leqslant T(n-1) + T(n-11) + p(n) = O\left(1.185^n\right)$$

Approximately pruning the search tree (5): works also for . . .

- MIN SET COVER (Bourgeois, Escoffier & P (2009))
- BANDWIDTH (Cygan & Pilipczuk (2010))
- MIN and MAX SAT (Escoffier, P & Tourniaire (2011))

Randomization

The key idea

Achieving ratio r with complexity better than $O\left(\gamma^{rn}\right)$

- Randomly split the graph into subgraphs in such a way that the problem at hand is to be solved in graphs G_i' of order $r'n$ with $r' < r$
- Compute the probability $\Pr[r]$ of an r-approximation
- Repeat splitting $N(r)$ times to bet r-approximation with probability ~ 1 (in time $N(r)\gamma^{r'n}$)

It works for MAX INDEPENDENT SET, MIN VERTEX COVER, MIN SET COVER, ...

Is subexponential approximation possible?

MIN COLORING

polynomially inapproximable within $\frac{\chi(G)+1}{\chi(G)}$ (Garey & Johnson (1979)) but **exponentially approximable** within $\frac{\chi(G)+1}{\chi(G)}$ ((Björklund, Husfeldt & Koivisto (2006)), (Bourgeois, Escoffier & P (2009)))

If it is subexponentially approximable within better than $\frac{\chi(G)+1}{\chi(G)}$, then MIN COLORING is solvable in subexponential time!!!

DISPROVAL OF THE *ETH* FOR MIN COLORING!!!!

Further questions: structure of moderately exponential approximation

- More tools proper to moderately exponential approximation
- Moderately exponential approximation preserving reductions?
- Is it possible to get inapproximability results?
  - Of what kind?
  - Under what complexity conditions?

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